ON CES PRODUCTION FUNCTION*

By KENJIRO ARA**

Let the production function be denoted by

(1)......... $Y=F(X_1, X_2, ..., X_S)$,

where $Y=$ output and $X_i=$ the *i-th* input. In what follows we assume that

(i) $\dfrac{\partial Y}{\partial X_i}>0 \qquad (i=1, 2, ..., s)$

(ii) $\alpha_i\left\{\dfrac{\partial Y}{\partial X_i}\right\}^\sigma X_i=\alpha_j\left\{\dfrac{\partial Y}{\partial X_j}\right\}^\sigma X_j \qquad (i, j=1, 2, ..., s)$,

where $\alpha_i>0$ and $\sigma\geqq 0$ are respectively constants. As is easily seen, assumption (ii) means that we are dealing with a CES (Constant Elasticity of Substitution) production function because of

$$\sigma=\frac{g\left\{\dfrac{X_j}{X_i}\right\}}{g\left\{\dfrac{\partial Y}{\partial X_i}\div\dfrac{\partial Y}{\partial X_j}\right\}},$$

where $g(x)\equiv\dfrac{dx}{x}$. Notice that the production function (1) is not necessary to be an homogeneous function, still less linear homogeneous. For the sake of convenience, we use the notation

$$\sigma=\frac{1}{1+\rho}.$$

Now we can prove the following

Theorem I: In case of $\rho\neq 0$ ($\therefore\ \sigma\neq 1$) *the production function can be always expressed by the form*

$$Y=\Psi(B\{\beta_1 X_1^{-\rho}+\beta_2 X_2^{-\rho}+\cdots+\beta_S X_S^{-\rho}\}),$$

where Ψ *is any differentiable function and* B *and* β_i ($\beta_i>0$ *and* $\Sigma\beta_i=1$) *are respectively constants.*

Proof: Assuming that $\rho\neq 0$, let X_i be transformed into Z_i in such a way that

(2)......... $Z_i=-\dfrac{1}{\gamma_i}\cdot\dfrac{1}{\rho}X_i^{-\rho} \qquad (i=1, 2, ..., s)$,

where $\gamma_i=\alpha_i^{1+\rho}$. From this, we have

(3)......... $\dfrac{dZ_i}{dX_i}=\dfrac{1}{\gamma_i}X_i^{-(1+\rho)} \qquad (i=1, 2, ..., s)$.

Because of (2), the production function can be rewritten by the equation

(4)......... $Y=f(Z_1, Z_2, ..., Z_S)$,

where f is any differentiable function. Totally differentiating (4), we have

---

* In writing this paper, the author owes very much to Prof. K. Yamada, S. Nabeya and S. Kataoka. The remaining errors which may exist, however, must be attributed solely to the author.

** Professor (*Kyōju*) of Economics.

(5)......... $dY=\frac{\partial Y}{\partial Z_1}dZ_1+\frac{\partial Y}{\partial Z_2}dZ_2+ .... +\frac{\partial Y}{\partial Z_S}dZ_S.$

Now let us notice that

(6)......... $\frac{\partial Y}{\partial X_i}=\frac{\partial Y}{\partial Z_i}\ \frac{\partial Z_i}{\partial X_i}=\frac{\partial Y}{\partial Z_i}\cdot\frac{1}{\gamma_i}\cdot X_i^{-(1+\rho)}$ $\quad (i=1, 2, ..., s),$

then we have

(7)......... $\frac{\partial Y}{\partial Z_i}=\gamma_i\frac{\partial Y}{\partial X_i}X_i^{1+\rho}=\gamma_j\frac{\partial Y}{\partial X_j}X_j^{1+\rho}=\frac{\partial Y}{\partial Z_j},$

or

(8)......... $\frac{\partial Y}{\partial Z_1}=\frac{\partial Y}{\partial Z_2}= \cdots =\frac{\partial Y}{\partial Z_S}=\frac{\partial Y}{\partial Z}.$

Putting (8) into (5), we get

(9)......... $dY=\frac{\partial Y}{\partial Z}\{dZ_1+dZ_2+ .... +dZ_S\}.$

For the moment let Y be given at $Y=Y_0$. Then $dY_0=0$. By assumption, $\frac{\partial Y}{\partial Z}>0$ for any level of Y. Therefore it follows

(10)......... $dZ_1+dZ_2+ .... +dZ_S=0.$

Integrating (10), we have

$$Z_1+Z_2+ .... +Z_S=c_0,$$

where c_0 is a constant of integration at $Y=Y_0$. In the same way we have

$$Z_1+Z_2+ .... +Z_S=c_i$$

at $Y=Y_i$, where Y_i is any given level of Y and c_i is a constant associated with $Y=Y_i$. Thus we have generally

$$Z_1+Z_2+ .... +Z_S=\phi(Y)$$

or

$$Y=\Psi(Z_1+Z_2+ .... +Z_S)$$

or

(11)......... $Y=\Psi(B\{\beta_1X_1^{-\rho}+\beta_2X_2^{-\rho}+ .... +\beta_SX_S^{-\rho}\}),$

where

$$\beta_i=\frac{\frac{1}{\gamma_i}}{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+ .... +\frac{1}{\gamma_S}}>0$$

$$B=-\frac{1}{\rho}\left\{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+ .... +\frac{1}{\gamma_S}\right\}.$$

Thus we could prove Theorem I. The case of $\sigma=1$ will be discussed in the connection with Theorem III.

Theorem II: In case of $\rho\neq 0$ ($\therefore\ \sigma\neq 1$), *the CES production function can be always expressed by the form*

$$Y=A\cdot[\beta_1X_1^{-\rho}+\beta_2X_2^{-\rho}+ .... +\beta_SX_S^{-\rho}]^{-\frac{m}{\rho}}$$

if it is a homogeneous function of m-th degree, where A is a constant.

Proof: For the sake of simplicity, let us put

(12)......... $X\equiv B[\beta_1X_1^{-\rho}+\beta_2X_2^{-\rho}+ .... +\beta_SX_S^{-\rho}],$

then we have $Y=\Psi(X)$. Because of homogeneity of m-th degree, we have

(13)......... $p^mY=\Psi(p^{-\rho}\cdot X),$

where p is any real number. Let $p^{-\rho}=X^{-1}$ ($\therefore$ $p^m=X^{\frac{m}{\rho}}$), then it follows

(14).........$X^{\frac{m}{\rho}}\cdot Y=\Psi(1)$

or

(15).........$Y=\Psi(1)X^{-\frac{m}{\rho}}$,

where $\Psi(1)$ is a constant. Thus we have finally

(16).........$Y=A\cdot[\beta_1X_1^{-\rho}+\beta_2X_2^{-\rho}+ .... +\beta_SX_S^{-\rho}]^{-\frac{m}{\rho}}$,

where

$$A=\Psi(1)\cdot B^{-\frac{m}{\rho}}=\text{constant}.$$

Theorem III: The production function (16) *can be expressed by the following equations*

(i) $$\lim_{\rho\to 0} Y=A^*\cdot X_1^{m\beta_1}\cdot X_2^{m\beta_2}.... \cdot X_S^{m\beta_S}$$

(ii) $$\lim_{\rho\to\infty} Y=A^{**}\min[X_1^m, X_2^m, ..., X_S^m],$$

where A^* and A^{**} are respectively constants.

Proof: From (16) it follows

(17).........$\log Y=\log A-m\dfrac{\log[\beta_1X_1^{-\rho}+\cdots+\beta_SX_S^{-\rho}]}{\rho}$.

Now let us first consider the case of $\rho\to 0$ ($\therefore$ $\sigma\to 1$), namely

(18).........$\lim\limits_{\rho\to 0}\log Y=\log A^*-\lim\limits_{\rho\to 0} m\dfrac{\log[\beta_1X_1^{-\rho}+\cdots+\beta_SX_S^{-\rho}]}{\rho}$,

where $\log A^*=\lim\limits_{\rho\to 0}\log A$. As is easily seen, the second term of the right side of (18) is reduced to $\dfrac{0}{0}$ (remember that $\log\Sigma\beta_i=\log 1=0$), then we can apply the l'Hospital's rule

(19).........$\lim\limits_{x\to a}\dfrac{h(x)}{f(x)}=\lim\limits_{x\to a}\dfrac{h'(x)}{f'(x)}$ for $f'(x)\neq 0$

to this case. Remembering that

$$\frac{d}{d\rho}m\log[\beta_1X_1^{-\rho}+ .... +\beta_SX_S^{-\rho}]=-m\frac{\beta_1X_1^{-\rho}\log X_1+\cdots+\beta_SX_S^{-\rho}\log X_S}{\beta_1X_1^{-\rho}+\cdots+\beta_SX_S^{-\rho}},$$

we have finally

(20).........$\lim\limits_{\rho\to 0}\log Y=\log A^*+m[\beta_1\log X_1+ .... +\beta_S\log X_S]$

or, taking antilogarithm,

(21).........$\lim\limits_{\rho\to 0} Y=A^*\cdot X_1^{m\beta_1}\cdot X_2^{m\beta_2}+ .... \cdot X_S^{m\beta_S}$.

This is obviously a Cobb-Douglasian produdtion function with homogeneity of m-th degree.

Next we consider the case of $\rho\to\infty$ ($\therefore$ $\sigma\to 0$), namely

(22).........$\lim\limits_{\rho\to\infty}\log Y=\log A^{**}-\lim\limits_{\rho\to\infty} m\dfrac{\log[\beta_1X_1^{-\rho}+\cdots+\beta_SX_S^{-\rho}]}{\rho}$,

where $\log A^{**}=\lim\limits_{\rho\to\infty}\log A$. Now that the second term of the right side of (22) is reduced to $-\dfrac{\infty}{\infty}$, we can again apply the l'Hospital's rule (19) to this case. Thus we have

(23).........$\lim\limits_{\rho\to\infty}\log Y=\log A^{**}+\lim\limits_{\rho\to\infty} m\dfrac{\beta_1X_1^{-\rho}\log X_1+\cdots+\beta_SX_S^{-\rho}\log X_S}{\beta_1X_1^{-\rho}+\cdots+\beta_SX_S^{-\rho}}$.

Without loss of generality let us assume that

$$X_1<X_j \qquad (j=2, 3, ..., s),$$

then we have

$$\lim_{\rho\to\infty} m\frac{\beta_1 X_1^{-\rho}\log X_1+\cdots+\beta_S X_S^{-\rho}\log X_S}{\beta_1 X_1^{-\rho}+\cdots+\beta_S X_S^{-\rho}}$$
$$=\lim_{\rho\to\infty} m\frac{\beta_1\log X_1+\beta_2\left(\frac{X_2}{X_1}\right)^{-\rho}\log X_2+\cdots}{\beta_1+\beta_2\left(\frac{X_2}{X_1}\right)^{-\rho}+\cdots}$$
$$=m\log X_1,$$

namely

$$\lim_{\rho\to 0} Y=A^{**}X_1^m.$$

Next let us assume that

$$X_1=X_2<X_j \qquad (j=3, 4, \ldots, s),$$

then we have

$$\lim_{\rho\to\infty} m\frac{\beta_1 X_1^{-\rho}\log X_1+\cdots+\beta_S X_S^{-\rho}\log X_S}{\beta_1 X_1^{-\rho}+\cdots+\beta_S X_S^{-\rho}}$$
$$=\lim_{\rho\to\infty} m\frac{\beta_1\log X_1+\beta_2\log X_2+\beta_3\left(\frac{X_3}{X_1}\right)^{-\rho}\log X_3+\cdots}{\beta_1+\beta_2+\beta_3\left(\frac{X_3}{X_1}\right)^{-\rho}+\cdots}$$
$$=\frac{m\beta_1}{\beta_1+\beta_2}\log X_1+\frac{m\beta_2}{\beta_1+\beta_2}\log X_2$$
$$=m\log X_1=m\log X_2,$$

namely

$$\lim_{\rho\to\infty} Y=A^{**}X_1^m=A^{**}X_2^m.$$

The procedure is the same in other cases. Thus we could prove that

(24).........$\lim\limits_{\rho\to\infty} Y=A^{**}\min\,[X_1^m, X_2^m, \ldots, X_S^m]$.

This is obviously a limitational production function with homogeneity of m-th degree.

(1965, Oct.)